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Propagation of acoustic waves in two waveguides coupled by perforations. I. Theory

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Abstract. — The problem of propagation in two guides coupled by perforations, important for a perforated tube muffler, is discussed. At low frequencies, if the distance between perforations is sufficiently large, a discrete model can be used. An exact equivalent circuit for a perforation is obtained by using modal theory and matricial formalism. A series inductance due to the existence of antisymmetrical field in the perforation is proved to exist, completing the perforation shunt impedance concept. This model is directly exploitable for lattice analysis. For homogeneous lattices (i.e. with identical propagation in the two guides), either regular or irregular, two modes exist: a planar mode, and a "flute" mode, either propagating or evanescent. Cutoff frequencies of periodic lattices are found to depend on either the shunt inductance or the series inductance, (the first cutoff depending on the shunt one). In inhomogeneous lattices, a new type of evanescent waves can exist, with non-zero energy flux, equal and opposite in sign in each guide. Finally, the effect of mean flow in such a lattice is discussed.

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List of symbols

a = height of the main guides (when $a_1 = a_2$)
a_1, a_2 = heights (2D case) or radii (circular case) of the main guides
A = coefficient of the matrix M (Eqs. (2, 29))
b = width (2D case) or radius (circular case) of the perforation
B = coefficient of the matrix M (Eqs. (2, 29))
c = speed of sound
C = coefficient of the matrix M (Eqs. (2, 29))
d = transverse dimension (2D case)
D = coefficient of the matrix M (Eqs. (2, 29))
$E^{\pm}(z)$ = propagation matrix
$G(\mathbf{r}, \mathbf{r}')$ = Green function of an infinite guide
$\mathbf{h}$ = subvector of the matrix H_s
$h_0 = H_{s00}$
H = modified radiation impedance matrix of a perforation into a guide (Eq. (12))
I = energy flux
$j = (-1)^{1/2}$
k = planar mode wavenumber
k_i = wavenumber of duct mode i
l = distance between two perforations of a lattice
ℓ_a, $\ell_s = \mathsf{L}_a/\rho$, L_s/ρ
L_a, L_s = antisymmetrical and symmetrical specific inductances (Eq. (2))
L_p = acoustic inductance of a perforation ($= Z_p/j\omega$)
M = Mach number
M = second-order matrix (Eq. (29))
$\mathcal{M}$ = fourth-order matrix (Eq. (29))
$p = P_0$
$p(\mathbf{r})$ = acoustic pressure at point $\mathbf{r}$
$\mathbf{P}$ = vector of the modal coefficients of the acoustic pressure
$\mathcal{P}$ = fourth-order transfer matrix (Eq. (31))
$\mathbf{q}^{\pm}$ = first column vector of the matrix $Q^{\pm}$
$Q^{\pm}$ = matrices defined in equation (8)
$\mathbf{r}$ = 3D vector of the coordinates of a point in a guide
S = cross section area of a guide or a perforation
t = time
T = second-order transfer matrix for the planar mode of a guide
$\mathcal{T}$, $\mathcal{T}_r$ = fourth-order matrices (Eqs. (30, 32))
u = volume velocity
$\mathbf{U}$ = vector of the modal coefficients of the acoustic velocity (multiplied by S)
v = planar mode velocity ($= \mathbf{U}_0/S$)
$\mathcal{V}$, $\tilde{\mathcal{V}}$ = fourth-order vectors (Eqs. (30, 32))
$\mathbf{w}$ = 2D vector of the transverse coordinates in a guide
$\mathbf{y}$ = 2D vector of the coordinates on the surface of a perforation
z = longitudinal coordinate in a guide
z_c = characteristic impedance of the planar mode in a guide

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Y_s = specific admittance of a perforation (eq. (2))
Z_c = diagonal matrix of the characteristic impedances of the duct modes
Z_p = acoustic perforation impedance (Eq. (3d))
Z_{AA} = radiation impedance matrix of a perforation into a guide (Eq. (8))
Z_a = specific impedance of a perforation (Eq. (2))
α = infinite vector (Eq. (11))
β = infinite vector (Eq. (11))
γ = ratio of cross section areas: $\gamma_i = S_i/(S_1 + S_2)$
Γ = dimensionless propagation constant of a lattice
δ_{ij} = Kronecker symbol
η_i = eigenvalue of the mode i in a guide
ρ = gas density
ψ = eigenmodes vector
ω = angular frequency

Superscripts

' distinction of the higher order modes in a guide or on a perforation
— trace of a matrix
± incoming (+) and outcoming (-) waves or modes
t transpose of a matrix

Subscripts

a corresponds to the antisymmetrical field in a perforation
A surface of a perforation (see Fig. 4)
o corresponds to the planar mode
L left side of a perforation
R right side of a perforation
s corresponds to the symmetrical field in a perforation

1. Introduction

The problem of wave propagation in two guides coupled by perforations has received attention because of the practical importance of perforated tube mufflers. The first theoretical treatment of this kind of mufflers was given by Sullivan and Crocker (1978): they used an experimentally measured value for the perforation impedance and proposed a continuous model, by considering a density of perforations in the wall. This model was developed by many authors, especially in order to deduce a decoupling analysis of two types of waves (see e.g. Jayaraman and Yam, 1981; Rao and Munjal, 1984; Bento Coelho, 1983); Peat, 1988). In particular Jayaraman and Yam (1981) tried to simplify the model in order to define the higher limit frequency where no radial velocity fluctuation occurs.

Discrete models have been relatively seldom used (see e.g. Sullivan, 1979; Brzózka, 1987). Nevertheless, they are more closely related with the actual shape of a perforated tube muffler. Moreover, periodic media analysis can be applied and lead to interesting results at middle and higher frequencies. This type of analysis is classical since Stewart and Lindsay (1930) for lumped elements of 1D-acoustical systems, or Benade (1960) and Keefe (1990) for musical woodwind instruments. Therefore the first aim of the present paper is to apply the discrete model to guides coupled by perforations. Moreover, a precise description of the effect of one perforation can be achieved by using the modal theory of waveguides, and the second aim of the paper is to adapt Keefe's approach (1982) of woodwind toneholes to perforations between two guides. We will show that this theory, modified by using a matricial formalism used for discontinuities in waveguides (Kergomard, 1991), leads directly to a very convenient discrete model. It allows one to prove that the perforation (shunt) impedance is in general not sufficient for describing a perforation.

The present paper is essentially theoretical and analytical in the discussion of physical phenomena. It is partly restricted to tubes without mean flow (but the lattice analysis is concerned with both cases of zero flow and non-zero flow) and without dissipation (but the dissipation could be easily taken into account, see, for higher order modes in tubes, Kergomard et al, 1988). Moreover, it is a linear theory (see for numerical, non linear methods, Chang and Cummings, 1988, Morel et al, 1991). In section 2, is derived the transfer matrix for a perforation between two guides. In sections 3 and 4, the definition of a lattice of two guides coupled by perforations is given and the propagation in such a lattice is discussed. In section 3, the lattice is homogeneous (the propagation is identical in the two guides), regular or irregular, in section 4, the lattice is inhomogeneous (and regular): as an example, mean flows with different Mach numbers occur in the two guides.

In a second paper, we will compare the results of the present theory to experiments for the case of a perforated tube muffler.

2. Transfer matrix for a perforation

2.1. *Form of the result*

In this section, we show that the fourth order matrix relating the planar modes in the two guides can be written in the following form:

$$\begin{pmatrix} \gamma_1\, p_{1L} + \gamma_2\, p_{2L} \\ \gamma_1\, v_{1L} + \gamma_2\, v_{2L} \\ p_{1L} - p_{2L} \\ v_{1L} - v_{2L} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix} \begin{pmatrix} \gamma_1\, p_{1R} + \gamma_2\, p_{2R} \\ \gamma_1\, v_{1R} + \gamma_2\, v_{2R} \\ p_{1R} - p_{2R} \\ v_{1R} - v_{2R} \end{pmatrix} \tag{1}$$

where $\gamma_i = S_i/(S_1 + S_2)$, S_i is the cross section of guide $i (i = 1, 2)$, p and v are the planar mode pressure and velocity, respectively, subscripts L and R correspond to left and right side of the perforation (see Fig. 1), A, B, C, D are the coefficients of a transfer matrix corresponding to the equivalent electrical circuit shown in Figure 2. This

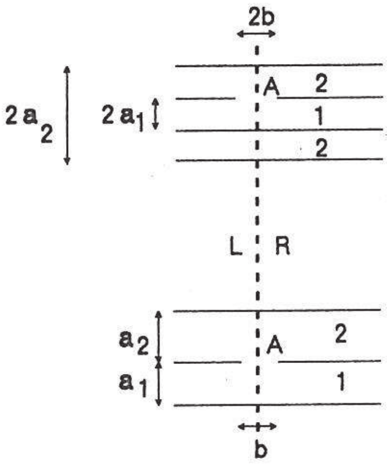

Figure 1. Two cases of two guides coupled by a perforation (1a: circular case, 1b: bidimensional, rectangular geometry).

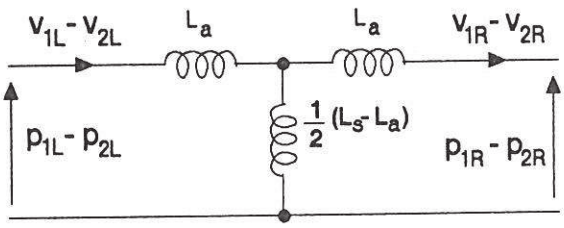

Figure 2. Equivalent electrical circuit for the matrix M of a perforation (pressure $\equiv$ voltage, velocity $\equiv$ current).

matrix is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{1 - \mathsf{Y}_s \mathsf{Z}_a} \begin{pmatrix} 1 + \mathsf{Y}_s \mathsf{Z}_a & 2\mathsf{Z}_a \\ 2\mathsf{Y}_s & 1 + \mathsf{Y}_s \mathsf{Z}_a \end{pmatrix} \tag{2}$$

where $\mathsf{Z}_a = j\omega \mathsf{L}_a, \mathsf{Y}_s = 1/j\omega \mathsf{L}_s$, and ω is the angular frequency.

The impedances Z_a and $1/\mathsf{Y}_s$ are ratios pressure/velocity, i.e. local impedances. The choice of the acoustic impedance (pressure/volume velocity) is excellent for the treatment of problems of discontinuities in waveguides, but in the present case, the specific impedance is more convenient, as shown by the lattice analysis (Sect. 3). In order to avoid confusion, we use large characters for the specific impedances (and admittances), and normal characters for the acoustic impedances.

The two inductances L_a and L_s correspond to the effect of the parts of the velocity field in the perforation that are antisymmetrical and symmetrical with respect to the plane $z = 0$ (see below in Sect. 2.4), respectively. These inductances vary slowly with the frequency when the transverse dimensions of the guides remain small compared to the wavelength.

This kind of result is similar to results obtained for similar junctions in electromagnetic waveguides (see especially Marcuvitz, 1948). Thus, we can find an equivalent circuit for the four-port network defined by equation (1), by adapting the results of this author. This circuit is shown in Figure 3: it is valid using the acoustic impedance analogy (pressure $\equiv$ voltage, volume velocity $\equiv$ current). Because of the duality of the rôles of pressure and velocity in equation (1), the same circuit is valid in the mechanical admittance (or mobility) analogy (velocity $\equiv$ voltage, force $\equiv$ current), by interverting Z_a and Y_s.

In the theoretical treatment of perforated tube mufflers (see e.g. Sullivan, 1979), the impedance Z_a is ignored, and the model of perforation is reduced to the following equations (we put $\mathsf{Z}_a = 0$ in equations (1) and (2)):

$$p_{1L} = p_{1R} = p_1, \quad p_{2L} = p_{2R} = p_2, \tag{3a}$$

$$S_1 (v_{1L} - v_{1R}) = -S_2 (v_{2L} - v_{2R}) = u \tag{3b}$$

$$p_1 - p_2 = Z_p u. \tag{3c}$$

As shown by circuit of figure 3, Z_p is the impedance of the perforation, related to Y_s by:

$$Z_p = \frac{1}{2} \left(\frac{1}{S_1} + \frac{1}{S_2} \right) \frac{1}{\mathsf{Y}_s}. \tag{3d}$$

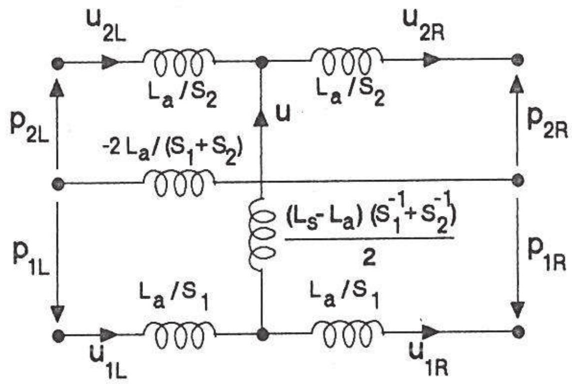

Figure 3. Equivalent electrical circuit of a perforation (pressure $\equiv$ voltage, volume velocity $\equiv$ current).

We show in section 2.4 that the volume velocity u is the planar mode volume velocity in the perforation.

The calculation of the result (1) can be made in three steps:

i) deriving the integral equation for matching one guide with the perforation (Sect. 2.2);
ii) calculating a modal expansion in the guides and the perforation, projecting the integral equation on control surfaces, and eliminating the higher modes in the guide (Sect. 2.3);
iii) connecting the two sets of equations for the two guides and eliminating the velocity in the perforation (Sect. 2.4).

2.2. *Basic integral equation*

In this section and the next section, we consider only one guide (without subscript) and the perforation. The surfaces L, R and A (A corresponding to the perforation, see Fig. 4) with the rigid walls delimit a volume.

For the moment, the abscissae of surfaces L and R are located at arbitrary positions z_L and z_R, respectively, where the z-coordinate is the waveguide axis. The treatment is identical to the treatment for a right-angle bifurcation in a guide. Several forms of the equation have been chosen: Lapin (1960) and Keefe (1982) considered a volume source distribution in the volume, corresponding to the planar mode existing when surface A is rigid. We prefer to consider no sources inside the volume, regarding the junction directly as a passive multiport. Thus, we write the following Helmholtz-Huyghens (see e.g. Bruneau, 1983 or Pierce, 1981) equation, valid for each point $\mathbf{r}$ in the volume (in the frequency domain):

$$p(\mathbf{r}) = \int_{S_A+S_L+S_R} [G(\mathbf{r},\mathbf{r}')\partial_{n'}p(\mathbf{r}') - \partial_{n'}G(\mathbf{r},\mathbf{r}')\, p(\mathbf{r}')]\, \mathrm{d}S'.$$

where ∂_n is the outward normal derivative to the surface. The restriction to the surfaces S_A, S_L and S_R is possible using an appropriate choice of the Green function.

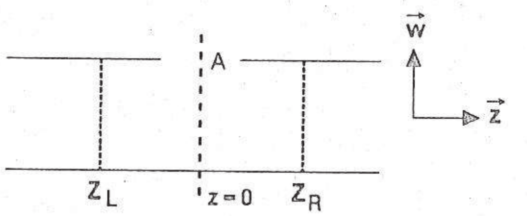

Figure 4. Control volume between surfaces A, L and R, and coordinates for a main guide.

The choice of the Green function of the closed cavity leads directly to an impedance matrix formulation, the second term of the integral vanishing (see Kergomard, 1992). Nevertheless, the decomposition of this function on the modes of the cavity gives a triple infinity of modes. We prefer to choose the Green function for the infinite guide, the number of modes being doubly infinite. Thus the second term of the integral does not vanish, but the integrals over S_L and S_R can be easily evaluated. As a matter of fact, if the pressure field inside the guide for $z \le z_L$ and $z \ge z_R$ is decomposed into two components, an incoming term $p_L^+(\mathbf{r})$ (from the left) and $p_L^-(\mathbf{r})$ (from the right) and an outcoming term $p_L^-(\mathbf{r})$ (to the left) and $p_R^+(\mathbf{r})$ (to the right), the values of the integral over S_L and S_R are simply $p_L^+(\mathbf{r})$ and $p_R^-(\mathbf{r})$. The explanation is as follows: if the guide is anechoïc for $z \le z_L$, the integral over S_L vanishes for p_L^- because $G(\mathbf{r},\mathbf{r}')$ and $p_L^+(\mathbf{r})$ can be interpreted as the contribution of sources located in $z_s \le z_L$. Because we choose the z-axis orientation to the right side (see Fig. 4), the same reasoning is valid for S_R and $p_R^-(\mathbf{r})$. Finally, the integral equation can be rewritten as follows:

$$p(\mathbf{r}) = \int_{S_A} G(\mathbf{r},\mathbf{r}')\partial_{n'}\, p(\mathbf{r}')\mathrm{d}S' + p_L^+(\mathbf{r}) + p_R^-(\mathbf{r}) \qquad (4)$$

(by definition, the derivative of the Green function of the infinite guide vanishes on S_A).

2.3. *Integral equation using modal expansion and matricial formalism*

We now decompose the field into modes of the guide. By omitting the time dependence $e^{j\omega t}(j = (-1)^{1/2})$, the pressure is written as follows:

$$p(\mathbf{r}) = \sum_i \psi_i(\mathrm{w})P_i(z)$$

where $r = (\mathrm{w}, z)$ (see Fig. 4), and ψ_i is a transverse mode solution of the equation $(\Delta_w + \psi_i)\psi_i = 0$ and the boundary condition $\partial_n\psi_i = 0$ on the walls. η_i is an eigenvalue, depending on the shape of the transverse cross-section of the guide. We choose the eigenmodes to satisfy the following orthogonality condition:

$$S^{-1}\int_S \psi_i\psi_j\, \mathrm{d}S = \delta_{ij}$$

where δ_{ij} is the Kronecker symbol and S is the cross-section of the guide, thus the modes are dimensionless.

Then we use a matricial formalism for the modal expansion (see Kergomard, 1991). $\mathbf{P}(z)$ and $\psi(\mathrm{w})$ are defined as column vectors, thus, if the superscript t indicates the transposed matrix of a given matrix (or of a vector):

$$p(\mathbf{r}) = {}^t\psi(\mathrm{w})\mathbf{P}(z) \qquad \mathbf{P}(z) = E^+(z)\mathbf{P}^+ + E^-(z)\mathbf{P}^- \quad (5)$$

where $\mathbf{P}^+$, $\mathbf{P}^-$ are column vectors and $E^\pm$ are diagonal matrices defined as follows:

$$E_{ij}^{\pm}(z) = e^{\mp jk_i z}\delta_{ij}$$

where k_i is the wavenumber of the mode i in the z-direction: $k_i^2 = k^2 - \eta_i^2$, $k = \omega/c$ is the planar mode wavenumber (for $i = 0$, $\eta_i = 0$), and c is the speed of sound. Equation (5) is valid on the left and right sides of the surface S_A, but the existence of the perforation implies that the coefficients are different on the two sides, and are noted $\mathbf{P}_L^{\pm}$ and $\mathbf{P}_R^{\pm}$, respectively. The problem is the derivation of a relation between these coefficients.

Using the matricial formalism, the orthogonality relation is written as follows:

$$S^{-1}\int_S \psi\, {}^t\psi \mathrm{d}S = \mathbb{1}$$

where $\mathbb{1}$ is the unit matrix .The Green function can be written as follows (see e.g. Bruneau, 1983 or Pierce, 1981):

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2j\omega\rho}\, {}^t\psi(\mathrm{w})Z_c E^-(z)E^+(z')\psi(\mathrm{w}') \qquad (6)$$

for $z' > z$

where ρ is the average air density, and Z_c the characteristic impedance (diagonal) matrix:

$$Z_{cij} = \frac{\rho c}{S}\frac{k}{k_i}\delta_{ij}. \tag{7}$$

This dimension of the acoustic impedance corresponds to the ratio of pressure to volume velocity (of course, for the higher order modes, the volume velocity is zero). The axial velocity is decomposed as follows, using the same matricial formalism:

$$v_z(\mathbf{r}) = S^{-1}\ {}^{\mathrm{t}}\psi(\mathbf{w})\mathbf{U}(z),$$

where

$$\mathbf{U}(z) = Z_c^{-1}\left(E^+(z)\mathbf{P}^+ - E^-(z)\mathbf{P}^-\right)$$

because $\partial_z p = -j\omega\rho v_z$.

It remains to decompose the field on the surface A. If the surface is planar , one can consider this surface as the cross section of a guide: this decomposition is classical for computing the effect of a diaphragm between two guides (see Vassallo, 1985). Thus if $\mathbf{y}$ is the 2D-vector of the coordinates of a point of the surface A, pressure $p_{\mathrm{A}}(\mathbf{y})$ and normal velocity $v_{\mathrm{A}}(\mathbf{y})$ to the surface are decomposed similarly to pressure $p(\mathbf{r})$ and axial velocity $v_z(\mathbf{r})$ at a given abscissa z of the guide:

$$p_{\mathrm{A}}(\mathbf{y}) = {}^{\mathrm{t}}\psi_{\mathrm{A}}(\mathbf{y})\mathbf{P}_{\mathrm{A}},\ v_{\mathrm{A}}(\mathbf{y}) = \frac{1}{S_{\mathrm{A}}}\ {}^{\mathrm{t}}\psi_{\mathrm{A}}(\mathbf{y})\mathbf{U}_{\mathrm{A}}$$

where ψ_{A} the eigenmodes vector of this surface, $\mathbf{P}_{\mathrm{A}}$ and $\mathbf{U}_{\mathrm{A}}$ two column vectors. If the surface is not planar (it is the case if guides 1 and 2 are concentric guides), this decomposition is more delicate, but we assume that it exists.

Now the projection of equation (4) can be made on the eigenmodes on S_{L}, S_{R}, and S_{A} respectively, by using the orthogonality relations, and one obtains:

$$\begin{aligned}
\mathbf{P}_{\mathrm{L}}^- &= -\frac{1}{2}Z_c\ {}^{\mathrm{t}}Q^+\mathbf{U}_{\mathrm{A}} + \mathbf{P}_{\mathrm{R}}^- \\
\mathbf{P}_{\mathrm{R}}^+ &= -\frac{1}{2}Z_c\ {}^{\mathrm{t}}Q^-\mathbf{U}_{\mathrm{A}} + \mathbf{P}_{\mathrm{L}}^+ \\
\mathbf{P}_{\mathrm{A}} &= -Z_{\mathrm{AA}}\mathbf{U}_{\mathrm{A}} + Q^+\mathbf{P}_{\mathrm{L}}^+ + Q^-\mathbf{P}_{\mathrm{R}}^-
\end{aligned} \tag{8}$$

where

$$Q^{\pm} = \frac{1}{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \psi_{\mathrm{A}}(\mathbf{y})\ {}^{\mathrm{t}}\psi(\mathbf{w})E^{\pm}(z)\mathrm{d}S$$

and

$$Z_{\mathrm{AA}} = \frac{j\omega\rho}{S_{\mathrm{A}}^2}\int_{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \psi_{\mathrm{A}}(\mathbf{y})G(\mathbf{y},\mathbf{y}')\ {}^{\mathrm{t}}\psi_{\mathrm{A}}(\mathbf{y}')\mathrm{d}S\mathrm{d}S'.$$

If we assume that the perforation is located sufficiently far from other perforations (or discontinuities, or sources), i.e. the distance between two perforations is larger than the diameter of the guide, we say that the discontinuities are decoupled. This condition is classical, and is discussed in the case of simple step discontinuities for example by Pierce (1981, p. 347) or Kergomard (1991). In the vectors $\mathbf{P}_{\mathrm{L}}^+$ and $\mathbf{P}_{\mathrm{R}}^-$ coming to the perforation, the evanescent modes elements can also be ignored. At low frequencies, only the planar mode propagates, and we write the two first equations (8) for this mode only, and modify the third equation:

$$\begin{aligned}
p_{\mathrm{L}}^- &= -\frac{1}{2}z_c\ {}^{\mathrm{t}}\mathbf{q}^+\mathbf{U}_{\mathrm{A}} + p_{\mathrm{R}}^- \\
p_{\mathrm{R}}^+ &= -\frac{1}{2}z_c\ {}^{\mathrm{t}}\mathbf{q}^-\mathbf{U}_{\mathrm{A}} + p_{\mathrm{L}}^+ \\
\mathbf{P}_{\mathrm{A}} &= -Z_{\mathrm{AA}}\mathbf{U}_{\mathrm{A}} + \mathbf{q}^+p_{\mathrm{L}}^+ + \mathbf{q}^-p_{\mathrm{R}}^-.
\end{aligned} \tag{9}$$

In these expressions, $p_{\mathrm{L}}^{\pm}$ and $p_{\mathrm{R}}^{\pm}$ are the first elements of the vectors $\mathbf{P}_{\mathrm{L}}^{\pm}$ and $\mathbf{P}_{\mathrm{R}}^{\pm}$, respectively, z_c is the characteristic impedance of the planar mode $(\rho c/S)$, and the vectors $\mathbf{q}^{\pm}$ are the first column of matrix $Q^{\pm}$.

We will now calculate the relation between the quantities $p = p^+ + p^-$, $u = (p^+ - p^-)/z_c$ for each side (subscripts L and R). These quantities correspond to the values of the pressure and volume velocity of the planar mode at the same abscissa, i.e. at the origin $z = 0$. If the origin is located at the center of the perforation, the modal decomposition (Eq. (5)) in the main guide is fictitious, because it is valid only outside of the hole location, but this method (proposed by Schwinger and Saxon, 1968; or Keefe, 1982) is of great practical interest for its simplicity. Thus we can re-arrange the equation (9):

$$\begin{aligned}
&p_{\mathrm{L}} - p_{\mathrm{R}} = jz_c\ {}^{\mathrm{t}}\boldsymbol{\beta}\mathbf{U}_{\mathrm{A}};\ u_{\mathrm{L}} - u_{\mathrm{R}} = {}^{\mathrm{t}}\boldsymbol{\alpha}\mathbf{U}_{\mathrm{A}}, \\
&\mathbf{P}_{\mathrm{A}} = -H\mathbf{U}_{\mathrm{A}} + \frac{1}{2}\boldsymbol{\alpha}(p_{\mathrm{L}} + p_{\mathrm{R}}) - \frac{1}{2}jz_c\boldsymbol{\beta}(u_{\mathrm{L}} + u_{\mathrm{R}})
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\boldsymbol{\alpha} &= \frac{1}{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \cos kz\ \psi_{\mathrm{A}}\mathrm{d}S \quad \text{and} \\
\boldsymbol{\beta} &= \frac{1}{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \sin kz\ \psi_{\mathrm{A}}\mathrm{d}S
\end{aligned} \tag{11}$$

$$\left(2\boldsymbol{\alpha} = \mathbf{q}^+ + \mathbf{q}^-,\ 2j\boldsymbol{\beta} = -\mathbf{q}^+ + \mathbf{q}^-\right).$$

The matrix H is obtained from Z_{AA} by the following relation:

$$H = Z_{\mathrm{AA}} - \frac{1}{2}z_c\left(\boldsymbol{\alpha}^{\mathrm{t}}\boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathrm{t}}\boldsymbol{\beta}\right).$$

Because $\boldsymbol{\alpha}^{\mathrm{t}}\boldsymbol{\alpha}$ can be written in the form:

$$\boldsymbol{\alpha}^{\mathrm{t}}\boldsymbol{\alpha} = \frac{1}{S_{\mathrm{A}}^2}\int_{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \cos kz\ \cos kz'\psi_{\mathrm{A}}(\mathbf{y})\ {}^{\mathrm{t}}\psi_{\mathrm{A}}(\mathbf{y}')\mathrm{d}S\mathrm{d}S'$$

and $\boldsymbol{\beta}^{\mathrm{t}}\boldsymbol{\beta}$ in a similar from, we obtain:

$$\begin{aligned}
&\boldsymbol{\alpha}^{\mathrm{t}}\boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathrm{t}}\boldsymbol{\beta} = \\
&= \frac{1}{S_{\mathrm{A}}^2}\int_{S_{\mathrm{A}}}\int_{S_{\mathrm{A}}} \psi_{\mathrm{A}}(\mathbf{y})\cos k(z - z')^{\mathrm{t}}\psi_{\mathrm{A}}(\mathbf{y}')\mathrm{d}S\mathrm{d}S'.
\end{aligned}$$

This term corresponds to the imaginary part of the planar mode term of the Green function, thus:

$$H = \frac{j\omega\rho}{S_A^2} \int_{S_A} \int_{S_A} \psi_A(y) \mathrm{Re}[G(y, y')]^t \psi_A(y') \mathrm{d}S \mathrm{d}S' \cdot \tag{12}$$

Because the real part of the Green function tends to a constant value when the frequency tends to zero, the matrix H is inductive, i.e. each term is proportionnal to $j\omega\rho$. Moreover, the vector β is inductive too when the frequency tends to zero, and α has only one non-zero element, corresponding to the planar mode of the surface A.

2.4. *Derivation of the transfer matrix*

Using equations (10) for the two guides 1 and 2, and eliminating the pressure vectors, we obtain:

$$\begin{aligned}
p_{1L} - p_{1R} &= j z_{c1}\, {}^t\beta \mathbf{U}_A \\
p_{2L} - p_{2R} &= -j z_{c2}\, {}^t\beta \mathbf{U}_A \\
u_{1L} - u_{1R} &= {}^t\alpha \mathbf{U}_A = -(u_{2L} - u_{2R}) \\
(H_1 + H_2)\mathbf{U}_A &= \frac{1}{2}\alpha (p_{1L} + p_{1R} - p_{2L} - p_{2R}) \\
&\quad - \frac{1}{2} j z_{c1} \beta (u_{1L} + u_{1R}) \\
&\quad + \frac{1}{2} j z_{c2} \beta (u_{2L} + u_{2R}) \cdot
\end{aligned}$$

In these equations, we consider the normal vector to the surface S_A oriented from guide 1 to guide 2. We point out that the vectors α and β are common to guides 1 and 2, because the acoustic fundamental mode is the planar one, and is common to guides of arbitrary shape. Re-arranging the above equations, we obtain:

$$\left\{ \begin{aligned} S_1 p_{1L} + S_2 p_{2L} &= S_1 p_{1R} + S_2 p_{2R} \\ S_1 v_{1L} + S_2 v_{2L} &= S_1 v_{1R} + S_2 v_{2R} \end{aligned} \right. \tag{13a}$$

$$\begin{aligned}
& p_{1L} - p_{2L} = p_{1R} - p_{2R} + j\rho c \left(S_1^{-1} + S_2^{-1}\right) {}^t\beta \mathbf{U}_A \\
& v_{1L} - v_{2L} = v_{1R} - v_{2R} + \left(S_1^{-1} + S_2^{-1}\right) {}^t\alpha \mathbf{U}_A \\
& (H_1 + H_2)\mathbf{U}_A = \frac{1}{2}\alpha (p_{1L} - p_{2L} + p_{1R} - p_{2R}) \\
& \qquad - \frac{1}{2} j \beta \rho c (v_{1L} - v_{2L} + v_{1R} - v_{2R}) \cdot
\end{aligned} \tag{13b}$$

The two first equations (13a) can be interpreted as the equilibrium of forces and conservation of volume velocities in $z = 0$, respectively, and correspond to the first (2×2) submatrix in equation (1). The other equations (13b), by eliminating the velocity profile $\mathbf{U}_A$ on the perforation, correspond to the (ABCD) matrix in equation (1). In the common case where the plane $z = 0$ is a plane of symmetry for the hole, it is possible to decompose the modes ψ_A of the hole (i.e. on the surface A) into symmetrical modes and antisymmetrical modes. Thus, the vectors and matrix can be decomposed as follows:

$$\begin{aligned}
& \mathbf{U}_A = \left(\frac{\mathbf{U}_{As}}{\mathbf{U}_{Aa}}\right); \ \alpha = \left(\frac{\alpha_s}{0}\right); \\
& \beta = \left(\frac{0}{\beta_a}\right); \ H_i = \left(\begin{array}{c|c} H_{is} & 0 \\ \hline 0 & H_{ia} \end{array}\right)
\end{aligned} \tag{14}$$

($i = 1$ or 2), and the equations become:

$$\begin{aligned}
p_{1L} - p_{2L} &= p_{1R} - p_{2R} + j\rho c \left(S_1^{-1} + S_2^{-1}\right) {}^t\beta_a \mathbf{U}_{Aa} \\
v_{1L} - v_{2L} &= v_{1R} - v_{2R} + \left(S_1^{-1} + S_2^{-1}\right) {}^t\alpha_s \mathbf{U}_{As} \\
(H_{1s} + H_{2s})\mathbf{U}_{As} &= \frac{1}{2}\alpha_s (p_{1L} - p_{2L} + p_{1R} - p_{2R}) \\
(H_{1a} + H_{2a})\mathbf{U}_{Aa} &= -\frac{1}{2} j \beta_a \rho c (v_{1L} - v_{2L} + v_{1R} - v_{2R}) \cdot
\end{aligned}$$

By eliminating $\mathbf{U}_{As}$ and $\mathbf{U}_{Aa}$, we obtain:

$$\begin{aligned}
p_{1L} - p_{2L} &= p_{1R} - p_{2R} + \mathsf{Z}_a (v_{1L} - v_{2L} + v_{1R} - v_{2R}) \\
v_{1L} - v_{2L} &= v_{1R} - v_{2R} + \mathsf{Y}_s (p_{1L} - p_{2L} + p_{1R} - p_{2R})
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\mathsf{Z}_a &= \frac{1}{2}\rho^2 c^2 \left(S_1^{-1} + S_2^{-1}\right)^t \beta_a (H_{1a} + H_{2a})^{-1} \beta_a \\
\mathsf{Y}_s &= \frac{1}{2}\left(S_1^{-1} + S_2^{-1}\right) {}^t\alpha_s (H_{1s} + H_{2s})^{-1} \alpha_s \cdot
\end{aligned} \tag{16}$$

Equation (15) corresponds to the matrix (ABCD) given in equation (2), and to the equivalent circuit shown in Figure 2. Thus the proof of equations (1) and (2) is achieved, the elements of the circuit being given by equation (16) (where α and β are given by Eq. (11) and H by Eq. (12)).

2.5. *Discussion of the result*

The inductive nature of both Z_a and Y_s when frequency tends to zero is due to the inductive character of β and H, α being a constant, as pointed out earlier. Moreover, at low frequencies, $k_i = j\eta_i \left[1 + 0\left(\frac{\omega^2}{c^2}/\eta_i^2\right)\right]$ if $i \neq 0$, and the expansion of the quantities Z_a and Y_s with respect to $(\omega/c\eta_i)^2$ and $(z\omega/c)^2$ exhibits the nature of the higher order terms: capacitances are in parallel with the inductances. This kind of expansion is discussed by Kergomard (1991).

Moreover, the value of Y_s given by equation (16) can be simplified by decomposing the symmetrical modes ψ_{As} of the hole into planar and and higher order modes, as follows:

$$\alpha_s = \left(\frac{\alpha_o}{\alpha'}\right) \quad H_s = H_{1s} + H_{2s} = \left(\begin{array}{c|c} h_0 & {}^t\mathbf{h} \\ \hline \mathbf{h} & H' \end{array}\right) \cdot$$

Thus, after some calculations, one obtains:

$$\mathsf{Y}_s = \frac{1}{2}\left(S_1^{-1} + S_2^{-1}\right) \times \left[\frac{\left(\alpha_0 - {}^t\alpha' H'^{-1}\mathbf{h}\right)^2}{h_0 - {}^t\mathbf{h}H'^{-1}\mathbf{h}} + {}^t\alpha' H'^{-1}\alpha'\right].$$

At low frequencies, $\alpha_0 = 1$, α' is proportional to ω^2, therfore the limit of Y_s is:

$$\mathsf{Y}_s = \frac{1}{2}\left(S_1^{-1} + S_2^{-1}\right) / \left(h_0 - {}^t\mathbf{h}H'^{-1}\mathbf{h}\right). \quad (17a)$$

With this decomposition, we obtain from equations (13b):

$$v_{1L} - v_{1R} = v_{2L} - v_{2R} + \left(S_1^{-1} + S_2^{-1}\right) u_A.$$

Using the conservation of volume velocities (see Eq. (13a)), one obtains equation (3b), with $u = u_A$. Thus u is verified to be the planar mode velocity in the perforation. Moreover, using the equation (3d), we see that the equation (17) leads to the following value of the shunt acoustic impedance of the perforation Z_p, (when the effect of the antisymmetrical modes is neglected):

$$Z_p = h_0 - {}^t\mathbf{h}H'^{-1}\mathbf{h} \quad (17b)$$

This expression is similar to the expression of a series impedance of a diaphragm of zero thickness between two guides (see Kergomard, 1991, Eq. (19)). We will not discuss further this similarity, but it allows to conclude that it is possible to obtain the exact result by using a double iteration calculation:

i) the perforation is considered as a limit of a guide of finite length, and the result is obtained by perturbing the result of a guide of infinite length;

ii) the result for a guide of infinite length is obtained by an iterative procedure from the plane piston approximation:

$$\mathsf{Y}_s = \frac{1}{2}\left(S_1^{-1} + S_2^{-1}\right)/h_0 \quad \text{or} \quad Z_p = h_0. \quad (18)$$

Thus the inversion of matrix H' is not needed in order to achieve the exact result. In a simpler way, it is possible to have a fine approximation of the result of equations (16) by using a variational formulation, as made by Keefe (1982). Consider, for example, a result of the form:

$$Y = {}^t\alpha H^{-1}\alpha, \quad \text{or} \quad Y = {}^t\alpha\mathbf{U}, \quad \text{with} \quad H\mathbf{U} = \alpha.$$

The quantity Y can be written as follows:

$$Y = \frac{\left({}^t\alpha\mathbf{U}\right)^2}{{}^t\mathbf{U}H\mathbf{U}}. \quad (19)$$

If there is an error ε in $\mathbf{U}$,

$$\begin{aligned} \mathsf{Y}(\varepsilon) &= \frac{\left[{}^t\alpha(\mathbf{U}+\varepsilon)\right]^2}{{}^t(\mathbf{U}+\varepsilon)H(\mathbf{U}+\varepsilon)} \\ &= \frac{\left({}^t\alpha\mathbf{U}\right)^2 + 2{}^t\alpha\varepsilon{}^t\mathbf{U}\alpha + \left({}^t\alpha\varepsilon\right)^2}{{}^t\mathbf{U}H\mathbf{U} + 2{}^t\varepsilon H\mathbf{U} + {}^t\varepsilon H\varepsilon}, \end{aligned}$$

thus

$$\begin{aligned} Y(\varepsilon) &= \frac{Y^2(0) + 2{}^t\alpha\varepsilon Y(0) + \left({}^t\alpha\varepsilon\right)^2}{Y(0) + 2{}^t\varepsilon\alpha + {}^t\varepsilon H\varepsilon} \\ &= \frac{Y(0)\left[Y(0) + 2{}^t\alpha\varepsilon\right] + \left({}^t\alpha\varepsilon\right)^2}{Y(0) + 2{}^t\alpha\varepsilon + {}^t\varepsilon H\varepsilon} \end{aligned}$$

thus $Y(\varepsilon) = Y(\text{o}) + O(\varepsilon^2)$ and equation (19) is a variational formulation for the result. Moreover, if $\mathbf{U}$ is multiplied by a scalar, equation (19) is unchanged.

If we apply this result to equation (16) for a plane profile of velocity $\mathbf{U}_{As}$, we show that the plane piston approximation (Eq. (18)) contains an error of second order of the velocity profile in the hole. If we apply the same result with a linear profile of velocity $v(z) = zx$ constant, we obtain an approximation for the series impedance Z_a:

$$\mathsf{Z}_a = \frac{\left({}^t\beta_a\mathbf{U}_{Aa}\right)^2}{{}^t\mathbf{U}_{Aa}\left(H_{1a} + H_{2a}\right)\mathbf{U}_{Aa}} \frac{1}{2}\rho^2 c^2 \left(S_1^{-1} + S_2^{-1}\right). \quad (20)$$

Of course, the same results can be obtained without matricial formalism, as Keefe (1982) made: an advantage is to avoid the modal decomposition in the perforation, but an exact resolution needs to solve an integral equation.

2.6. *Calculation of the inductances*

The rigorous evaluation of the inductances L_a and L_s is possible only for a few geometries. If the guides are rectangular and the perforation rectangular or circular, the complete calculation is possible. We first study the bidimensional geometry, and compare the plane piston approximation to the conformal transformation results, and then we study the circular , concentric geometry. We restrict the study to the range of frequencies where the wavelength is much larger than the transverse dimensions of the guides and the size of the perforation.

2.6.1. *Bidimensional, rectangular geometry*

From the equation (17b), the value of the (symmetrical) impedance $Z_p = j\omega L_p$ can be calculated. Nevertheless, the value when frequency tends to zero can be obtained directly from conformal transformation. The result, obtained by Khettabi (1994), is the following:

$$L_p d\pi/\rho = -2\,\text{Ln}\,\sinh\sigma + \frac{\left(a_1 - a_2\right)^2}{2a_1a_2}\,\text{Ln}\,\frac{\cosh\delta}{\cosh\sigma} - \text{Ln}\,\frac{4a_1a_2}{\left(a_1 + a_2\right)^2} \quad (21a)$$

where d is the common transverse dimension of the guides and the perforation, a_1 and a_2 the heights of the guides 1 and 2 (see Fig. 1b). The quantities σ and δ are defined as follows:

$$\sigma = (x_1 + x_2)/2 \qquad \delta = (x_1 - x_2)/2$$

where x_1 and x_2 are solutions of the equations:

$$\left\{\begin{array}{lcl} b\pi/2 & = & a_1 x_1 + a_2 x_2 \\ a_1 \sinh x_1 & = & a_2 \sinh x_2 . \end{array}\right. \quad (21b)$$

b is the length of the perforation , it is positive thus x_1 and x_2 are positive too. When the two guides are of equal height ($a_1 = a_2 = a$), $x_1 = x_2$ and the equation (21a) becomes:

$$L_p d\pi/\rho = -2 \text{ Ln } \sinh\,(b\pi/4a) \quad (22)$$

This equation was already given by Marcuvitz (1948) for electromagnetic waveguides.

Concerning the plane piston approximation, $Z_p = h_0 = h_{01} + h_{02}$, we obtain from the definition of the matrix H for each guide (see Eq. (12)):

$$h_{0i} = \frac{j\omega\rho}{d\pi} \left[-\frac{y_i}{6} + \frac{2}{y_i} \sum_{p=1}^{+\infty} \left[\frac{1}{p^2} + \frac{1}{y_i} \frac{e^{-y_i p} - 1}{p^3} \right] \right] \quad (23a)$$

where $i = 1$ or 2 and $y_i = b\pi/a_i$. Using a series expansion of Euler-Mac Laurin, this formula becomes:

$$\begin{aligned} h_{0i} \;=\; & \frac{j\omega\rho}{d\pi} \left[-\text{Ln } y_i + \frac{3}{2} - \frac{y_i^2}{144} + \frac{y_i^4}{43200} \right. \\ & \left. - \frac{y_i^6}{5080200} + \frac{y_i^8 10^{-8}}{4.35456} + O\left(y_i^{10}\right) \right] \end{aligned} \quad (23b)$$

Thus the plane piston approximation for small perforations is:

$$\begin{aligned} L_p d\pi/\rho = & - \text{ Ln } (b\pi/a_1) - \text{Ln } (b\pi/a_2) \\ & + 3 - \frac{b^2\pi^2}{144} \left(a_1^{-2} + a_2^{-2}\right) + O\left\{(b/a_i)^4\right\} \end{aligned} \quad (24)$$

It is interesting to compare this result with the series expansion of the exact equation (21):

$$\begin{aligned} L_p d\pi/\rho \;=\; & - \text{ Ln } (b\pi/a_1) - \text{Ln } (b\pi/a_2) \\ & + 4 \text{ Ln } 2 - \frac{1}{3}\frac{b^2\pi^2}{32} \left(a_1^{-2} + a_2^{-2}\right) \\ & + O\left\{b^4 \left(a_1^{-1} + a_2^{-1}\right)^4\right\} \end{aligned} \quad (25)$$

This result shows that to this order of expansion, the value of the symmetrical perforation inductance L_p is the average of the results obtained for the cases where the two guides have the same height, a_1 or a_2, respectively. Of course, this property is valid for the plane piston impedance too, because this impedance is the summation of the two radiation impedances of a piston into the guides 1 and 2.

Moreover, we can notice that the plane piston result is slightly larger than the exact result, as for the problem of the diaphragm of zero thickness in a guide (see for a discussion Pierce, 1981 or Kergomard and Garcia, 1987). The difference between the two problems of a perforation between the walls of two guides and a diaphragm inserted between two guides perpendicularly to the walls can be viewed as a difference of angle of incidence (tangential and normal, respectively) of a plane wave. If we restrict the discussion to guides of equal height, the difference in the result can be seen by replacing the function $\sinh(b\pi/4a)$ by the function $\sin(b\pi/2a)$ in equation (22). The lowest order terms are identical in the series expansion for small holes ($-\text{Ln}(b/a)$), thus the effect of the incidence is negligible for small holes.

The Figure 5 shows, when $a_1 = a_2$, the exact result (Eq. (22)) and the results of the plane piston approximation (Eq. (23a) and its expansion (23b)). We notice that the result can be either positive or negative, and vanishes for $b/a = 1.12$.

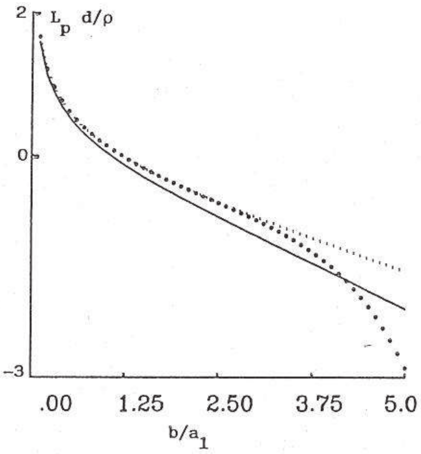

Figure 5. Dimensionless inductance ($L_p d/\rho$) for the 2D case with respect to the ratio b/a((—): conformal transformation result, Eq. (22), (∘∘∘): plane piston approximation, series expansion, Eq. (23b), (···): plane piston approximation, complete calculation, Eq. (23a)).

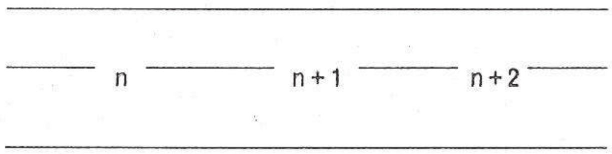

Figure 6. Lattice of guides coupled by perforations.

The discussion concerning the asymmetrical inductance L_a will be done more briefly. The result of the conformal transformation is:

$$L_a\pi/\rho = (a_1 + a_2) \text{ Ln } \frac{\cosh\,\delta}{\cosh\,\sigma} \quad (26)$$

with the notations of the equation (21b). An important result is that the inductance L_a is always negative. This fact can be interpreted as follows: if we consider the guide 2 closed at its ends, when frequency tends to zero, the existence of L_a is due for the incompressible flow in the guide 1 to the existence of a (large) cavity coupled to guide 1. Thus there is an enlargment of the flow lines, and a decreasing of the total kinetic energy (see for this kind of discussion Nederveen, 1963).

If the inductance L_a is compared with the symmetrical inductance L_s for small perforation, the latter tends to infinity and therefore L_a is negligible with respect to L_s.

A detailed study of the values of the inductances when the frequency increases can be found in Khettabi (1994).

2.6.2. *Circular, concentric geometry*

In the most common cases of mufflers, the guides and the perforation are circular. If the guides are concentric, the complete calculation is in principle possible, using Bessel functions of first and second kind, but the perforation surface is not plane, and the modal expansion on it is difficult. As discussed by Keefe (1982), when the radius b of the perforation is smaller than the radius a_1 of the inner guide (with subscript 1, see Fig. 1), one can use a small angle approximation and solve the problem in an approximate way.

In the present paper, we will restrict the study to the evaluation of an order of magnitude for the results for the case of small perforations $b \ll a_1$ and $b \ll a_2$. Thus we can use the variational form and consider that the effect of the outer guide (matrix H_2) is of the same order of magnitude as the effect of the inner guide (matrix H_1), as in the 2D case (see Sect. 2.6.1). Then we can use the calculations made by Keefe (1982) for a circular hole on a circular guide. By using the profiles $v_A = 1$ and $v_A = z$ for the calculation of Y_s and Z_a respectively (Eqs. (18) and (20)), one obtains at low frequencies $\boldsymbol{\alpha} = {}^t(1|0)$:

$$h_0 = 2h_{01} = \frac{j\omega\rho}{2b}\left(1 - 0.74\left(\frac{b}{a_1}\right)^2\right)$$

$$\frac{({}^t\boldsymbol{\beta}_a \mathbf{U}_{Aa})^2}{{}^t\mathbf{U}_{Aa} H_{1a} \mathbf{U}_{Aa}} = -\frac{S_1}{\rho c} j\ 0.728\ kb\ \frac{b}{a_1}$$

Thus

$$\mathsf{L}_s = \frac{\rho}{b}\frac{1}{(S_1^{-1} + S_2^{-1})}\left(1 - 0.74\left(\frac{b}{a_1}\right)^2\right) \tag{27}$$

and

$$\mathsf{L}_a = -0.57\ \rho b^2 a_1 \left(S_1^{-1} + S_2^{-1}\right) \cdot \tag{28}$$

The value of h_o is the classical value for the perforation impedance Z_p given by Rayleigh (1877) for a plane wave through apertures in a plane screen (see the discussion by Piercet, 1981) or by Fock (1941) for a diaphragm of zero thickness in a circular guide (see the discussion by Kergomard and Garcia, 1987). As for the 2D-case, the result does not depend on the incidence of a plane wave on the hole. Nevertheless, the dependance on the ratio b/a_1 given by Keefe (1982) for a side hole of a wind instrument (see Eq. (27)) is different from the (exact) dependance given by Fock for a diaphragm in a circular guide: in the first case, there is no term of first order, in the second, there is no term of second order. It is difficult to know if this difference is due to the difference of incidence for the wave or to the approximations that we made.

The order of magnitude of the ratio $\mathsf{L}_a/\mathsf{L}_s$ is therefore:

$$\frac{\mathsf{L}_a}{\mathsf{L}_s} = -0.57\ b^3\ a_1 \left(S_1^{-1} + S_2^{-1}\right)^2.$$

If $S_2 > S_1$, this ratio is less than $0.05\ (b/a_1)^3$. As Keefe (1982) wrote, it is in general very small, and it is often possible to neglect the product $\mathsf{Y}_s\mathsf{Z}_a$ in equation (2).

For the analysis of the lattice, it is useful to compare the characteristic lengths $\ell_a (= \mathsf{L}_a/\rho)$ and $\ell_s (= \mathsf{L}_s/\rho)$ with the distance ℓ between two perforations. In practice, one often encounters the case where $b < \ell < a_1$, and $S_1 < S_2$, thus:

$$-\frac{\ell_a}{\ell} = 0.57\frac{b^2 a_1}{\ell}\left(\frac{1}{S_1} + \frac{1}{S_2}\right) < 0.36\frac{b^2}{\ell a_1} \ll 1$$

and $$\frac{\ell}{\ell_s} \simeq \ell b\left(\frac{1}{S_1} + \frac{1}{S_2}\right) < \frac{2\ell b}{\pi a_1^2} \ll 1.$$

3. Homogeneous lattice analysis

3.1. *General properties of a homogeneous lattice*

We study now a lattice, i.e. a discrete medium: the acoustic field is considered only at certain values of the abscissa , we whoose for example the middle of the perforations. We are interested in *infinite lattice modes*, and mainly in their propagation constant Γ, a nondimensional quantity. These lattice modes are distinguished from the duct modes , used largely in the two first sections for the main guides and for the perforation surface.

We consider firstly a homogeneous lattice (see Fig. 6), i.e. two tubes with the same sound speed. The fourth-order matrix $\mathcal{M}$ of equation (1) can be written by using the second-order submatrices:

$$\mathcal{M} = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & M \end{array}\right) \quad \text{where} \quad M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{29}$$

This matrix corresponds to the effect of a perforation. If the propagation of the planar mode in the two guides is described by a classical transfer matrix T, identical for the two tubes, then the same matrix relates the two sub-vectors defined in equation (1):

$$\mathcal{V}(z) = \mathcal{T}\mathcal{V}(z + \ell) \tag{30}$$

where $\mathcal{T} = \left(\frac{T}{0}\middle|\frac{0}{T}\right)$,

$$\mathcal{V} = \begin{pmatrix} \gamma_1 p_1 + \gamma_2 p_2 \\ \gamma_1 v_1 + \gamma_2 v_2 \\ p_1 - p_2 \\ v_1 - v_2 \end{pmatrix}$$

and $$T = \begin{pmatrix} \cos k\ell & \rho c j \sin k\ell \\ \rho^{-1} c^{-1} j \sin k\ell & \cos k\ell \end{pmatrix}$$

$(\gamma_i = S_i/(S_1 + S_2))$.

Thus, for a succession of perforations (matrix M_i) and tubes of length ℓ_i, the transfer matrix is given by:

$$\mathcal{P} = \left(\begin{array}{c|c} \prod_i T_i & 0 \\ \hline 0 & \prod_i M_i T_i \end{array}\right). \tag{31}$$

Thanks to the choice of basic vector $\mathcal{V}$, the two modes of the lattice appear directly in this equation. They are:
i) the planar mode, for which pressure and velocity are identical in the two guides, propagates without effect of the perforations.
ii) the second mode, for which the forces $S_i p_i$ and volume velocities $S_i v_i$ are opposite in sign, is similar with a mode in a simple tube with open side holes. We can call it the "flute mode", by analogy with the propagation in wind instruments. As a matter of fact, for a tube with open side holes, we can consider the vector $(p,\ v)$, where p and v are the pressure and velocity of the planar duct mode. Then a lattice of open holes (from hole 1 to hole n) is described by the equation:

$$\begin{pmatrix} p \\ v \end{pmatrix}_1 = \prod_i M_i T_i \begin{pmatrix} p \\ v \end{pmatrix}_n .$$

The matrix M in this case has exactly the same form as in equation (2), but the shunt impedance includes the input impedance of a hole (see Benade, 1960 for a theory without antisymmetrical term Z_a, and Keefe, 1990 for a more complete theory).

These modes exist in either a regular or an irregular lattice. In the case of a regular lattice, the flute mode can be either propagating or evanescent, as analyzed by Benade (1960) and, with more details, in section 3.2.

Finally, we notice that it is useful to know the transfer matrix $\mathcal{P}$ for the vectors $\tilde{\mathcal{V}} = {}^t(p_1, v_1, p_2, v_2)$ in order to introduce boundary conditions. Thus, we need to know the transformation matrices $\mathcal{T}_r$ from $\mathcal{V}$ to $\tilde{\mathcal{V}}$. Using the submatrices calculation, we obtain easily:

$$\tilde{\mathcal{V}} = \mathcal{T}_r\, \mathcal{V}, \qquad \text{where}$$

$$\mathcal{T}_r = \left(\begin{array}{c|c} 1\!\!1 & \gamma_2 \\ \hline 1\!\!1 & -\gamma_1 \end{array} \right) \quad \text{and} \quad \mathcal{T}_r^{-1} = \left(\begin{array}{c|c} \gamma_1 & \gamma_2 \\ \hline 1\!\!1 & -1\!\!1 \end{array} \right). \tag{32}$$

3.2. *Periodic, homogeneous lattice: stop and pass bands of the flute mode*

3.2.1. *Propagation constant and characteristic impedance of the flute mode*

In the case of a regular lattice, the transfer matrix of a cell (see Fig. 6) is given for the flute mode by:

$$\begin{pmatrix} p_{1L} - p_{2L} \\ v_{1L} - v_{2L} \end{pmatrix}_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times$$

$$\begin{pmatrix} \cos k\ell & \rho c j \sin k\ell \\ \rho^{-1} c^{-1} j \sin k\ell & \cos k\ell \end{pmatrix} \begin{pmatrix} p_{1L} - p_{2L} \\ v_{1L} - v_{2L} \end{pmatrix}_{n+1}$$

For the definition and properties of a periodic 1D-lattice, we refer to Brillouin and Parodi (1956), and for acoustic lattices to Benade (1960), Depollier et al. (1990), Keefe (1990), Holland and Fahy (1991). Using equation (2), we obtain the dispersion equation:

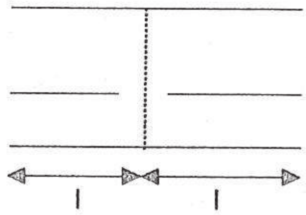

Figure 7. Symmetrical cell of a periodic lattice.

$$\cosh \Gamma = \frac{1 + \mathsf{Y}_s \mathsf{Z}_a}{1 - \mathsf{Y}_s \mathsf{Z}_a} \cos k\ell + \frac{j \sin k\ell}{1 - \mathsf{Y}_s \mathsf{Z}_a} \left(\frac{\mathsf{Z}_a}{\rho c} + \mathsf{Y}_s \rho c \right) \tag{33}$$

where Γ is the propagation constant. If Γ is real, the mode is evanescent, if Γ is imaginary, the mode propagates.

In order to exhibit the cutoff frequencies ($\cosh \Gamma = \pm 1$), we rewrite the equation as follows:

$$\cosh \Gamma = 1 + 2 \frac{\left(\mathsf{Y}_s \cos k\ell' + j \sin k\ell'/\rho c\right)\left(\mathsf{Z}_a \cos k\ell' + j\rho c \sin k\ell'\right)}{1 - \mathsf{Y}_s \mathsf{Z}_a} \tag{33a}$$

or

$$\cosh \Gamma = -1 + 2 \frac{\left(\mathsf{Y}_s j \sin k\ell' + \cos k\ell'/\rho c\right)\left(\mathsf{Z}_a j \sin k\ell' + \rho c \cos k\ell'\right)}{1 - \mathsf{Y}_s \mathsf{Z}_a} \tag{33b}$$

where $\ell' = \ell/2$.

An interesting result is the dependance of the cutoff frequencies on only one parameter, Y_s or Z_a. This result can be obtained in a more direct way by considering a symmetrical cell (see Fig. 7) divided into two half-cells. The left half-cell corresponds to the following transfer matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{j\omega(\mathsf{L}_s - \mathsf{L}_a)} \left(\cos k\ell' j\omega \mathsf{L}_s + \rho c\, j \sin k\ell'\right) & \cos k\ell' j\omega \mathsf{L}_a + \rho c j \sin k\ell' \\ \frac{1}{j\omega(\mathsf{L}_s - \mathsf{L}_a)} \left(\frac{j\omega \mathsf{L}_s}{\rho c} j \sin k\ell' + \cos k\ell'\right) & \cos k\ell' + \frac{j\omega \mathsf{L}_a}{\rho c} j \sin k\ell' \end{pmatrix} \tag{34}$$

and the right half-cell to the matrix $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Thus the transfer matrix of the symmetrical matrix is:

$$\begin{pmatrix} ad + bc & 2ab \\ 2cd & ad + bc \end{pmatrix} \qquad \text{(where } ad - bc = 1\text{)}.$$

The propagation constant can be deduced by:

$$\cosh \Gamma = ad + bc \qquad \text{or} \qquad \sinh \Gamma = 2\,(abcd)^{1/2}.$$

and the cutoff frequencies are given by:

$$a = 0 \quad \text{or} \quad d = 0 \rightarrow \cosh\Gamma = -1$$
$$b = 0 \quad \text{or} \quad c = 0 \rightarrow \cosh\Gamma = 1.$$

The characteristic impedance of the same cell is

$$Z_c = \left(\frac{ab}{cd}\right)^{1/2} \tag{35}$$

3.2.2. *Relation with the continuous model ; first cutoff frequency*

By calculating a series expansion with respect to $k\ell$ (by assuming the same order of magnitude for ℓ, ℓ_a, ℓ_s), we obtain from equation (33):

$$\cosh\Gamma = \frac{1+\ell_a/\ell_s+\ell/\ell_s}{1-\ell_a/\ell_s} - \frac{k^2\ell^2}{2}\frac{1+\ell_a/\ell_s+\ell/3\ell_s+2\ell_a/\ell}{1-\ell_a/\ell_s} \tag{36}$$

and for small ℓ_a/ℓ_s and ℓ/ℓ_s (see discussion below)

$$\Gamma^2 = 4\frac{\ell_a/\ell_s+\ell/2\ell_s}{1-\ell_a/\ell_s} - k^2\ell^2\frac{1+\ell_a/\ell_s+\ell/3\ell_s+2\ell_a/\ell}{1-\ell_a/\ell_s}$$

If $\beta = -j\Gamma/\ell$ and $n = 1/\ell$ are the wavenumber and number of holes per unit length, respectively, we obtain for $\ell_a = 0$ and $\ell/\ell \ll 1$:

$$\beta^2 = -\frac{2n}{\ell_s} + k^2$$

or, if $\dfrac{1}{\ell_s} = \dfrac{\rho}{2L_p}\left(\dfrac{1}{S_1}+\dfrac{1}{S_2}\right)$,

$$\beta^2 = -\frac{n\rho}{L_p}\left(\frac{1}{S_1}+\frac{1}{S_2}\right) + k^2, \text{ where } L_p = Z_p/j\omega \tag{37}$$

(see Eq. (3d))

This formula is given by Pierce (1981, p.356). If one uses the normalized specific impedance of the wall of the perforated tube, $\zeta = Z_p\ell 2\pi a_1/\rho c$, one obtains alternatively the formula given by Jayaraman and Yam (1981). This impedance can be expressed with respect to the percentage open area (porosity), thus the existence of several holes located at the same abscissa can be taken into account (see e.g. Crocker and Sullivan, 1978). We do not discuss this point further here, because a complete theory like the present one needs to include the interaction effects between holes. This problem for the case of a plane wave incident on a perforate panel is discussed by many authors (see e.g. Ingard, 1953; Nesterov, 1959; Rschevkin, 1963; Morfey, 1969; Melling, 1973; Leppington and Levine, 1973; Allard, 1993; Mechel, 1989).

We notice that for sufficiently large space ℓ between holes, the continuous equation (37) can be invalid, and that it omits the effect of antisymmetrical modes in the perforation. Nevertheless, this equation is often acceptable in practice, because the ratio ℓ_a/ℓ_s is very small (see Sect. 2.5) and the ratio ℓ/ℓ_s is small too. Concerning the first cutoff frequency itself, it does not depend on ℓ_a as explained now: the solutions of $\cosh\Gamma = 1$ (see Eq. (33a)) are solutions of:

$$\tan\frac{k\ell}{2} = -k\ell_a \tag{38a}$$

and

$$\tan\frac{k\ell}{2} = 1/k\ell_s \tag{38b}$$

Because $-2\ell_a/\ell$ is small, the root of equation (38a) satisfies $k > 2\pi/\ell$. Thus the root of equation (38b) is the first cutoff frequency:

$$k = \sqrt{\frac{2}{\ell\ell_s}}\left(1 - \frac{1}{12}\frac{\ell}{\ell_s} + O\left(\frac{\ell}{\ell_s}\right)^2\right) \tag{39}$$

At the zeroth order in ℓ/ℓ_s, the result is given by the continuous model (Eq. (37)).

3.2.3. *Input and output of a lattice in stop bands*

In stop bands, for a sufficiently large number of cells, the input and output impedances for the flute mode are their characteristic impedances. It is interesting to find an equivalent circuit for the input (for example), relating the plane modes in guides 1 and 2 and the plane mode in the lattice. Thus the input of the lattice will be analogous to the bifurcation of a large guide into two parallel guides, as treated by some authors (Miles, 1947; Bailin, 1951; Kergomard, 1991).

The problem is to relate the plane mode vectors $\begin{pmatrix} p_1 \\ S_1v_1 \end{pmatrix}_L$ and $\begin{pmatrix} p_2 \\ S_1v_2 \end{pmatrix}_L$ at the left of the first perforation to the plane mode vector at the input of the lattice, defined as

$$\begin{pmatrix} p \\ u \end{pmatrix}_L = \begin{pmatrix} \gamma_1 p_1 + \gamma_2 p_2 \\ S_1v_1 + S_2v_2 \end{pmatrix}_L.$$

We notice that we consider the volume velocities, instead of particle velocities. The definition of the vector $(p,\ u)$ gives two equations, and the knowledge of the input impedance Z_f of the flute mode leads to the third equation:

$$p_{1L} - p_{2L} = Z_f\,(v_{1L} - v_{2L})\,.$$

Thus, by omitting the subscript L:

$$p_1 - \frac{Z_f}{S_1}S_1v_1 = p_2 - \frac{Z_f}{S_2}S_2v_2 = p - \frac{Z_f}{S_1+S_2}u.$$

The equivalent circuit is deduced from this result, and shown in Figure 8. The result is similar to the result for a bifurcation, if one replaces $-Z_f/(S_1+S_2)$ by Z_{12}, where Z_{12} is the mutual discontinuity impedance of the two guides. Moreover, at low frequencies, Z_f is inductive (and Z_{12} too). From equation (35), by subtracting the inductance corresponding to the length $\ell' = \ell/2$, one obtains:

$$Z_f = Z_c - j\omega\rho\ell' = j\omega\rho\ell' \left[(1+\ell_s/\ell')^{1/2}(1+\ell_a/\ell')^{1/2} - 1\right]. \tag{40}$$

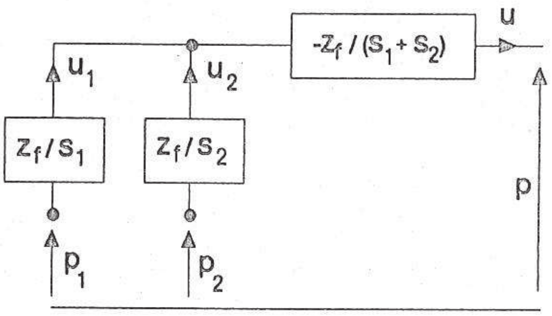

Figure 8 Equivalent circuit for the input of a periodic lattice in stop bands.

For this kind of equivalent circuit, Miles (1947) remarked that the relation between the three plane modes is equivalent to a translation of the bifurcation plane of a length $Z_f/j\omega\rho$, the equivalent circuit becoming very simple (the circuit of Fig. 8 with $Z_f = 0$). This is a generalization of the concept of length correction well known for the case of a simple step discontinuity, or for the case of the input of a tube provided with regular open side holes. Nevertheless, this equivalence with a translation of the bifurcation plane is submitted to certain conditions on the impedance p/u (for a discussion of this problem see Kergomard and Garcia, 1987).

This result will be important for analyzing the case of perforated tube mufflers at low frequencies, shown to be similar to an expansion chamber (with or without extended inlet and outlet). This question will be discussed in the companion paper.

4. Inhomogeneous regular lattice

4.1. *Transfer matrix for an inhomogeneous lattice*

If T_1 and T_2, the transfer matrices in each guide, are different, the fourth order matrix for a cell of lattice is given by :

$$\mathcal{P} = \mathcal{T}_r \mathcal{M} \mathcal{T}_r^{-1} \mathcal{T}$$

where $\mathcal{M}$ and $\mathcal{T}_r$ are given by the equations (29) and (32), respectively, and $\mathcal{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$.

Thus,

$$\mathcal{P} = \begin{pmatrix} (\gamma_1 + \gamma_2 M) T_1 & \gamma_2(\mathbb{1} - M) T_2 \\ \gamma_1(\mathbb{1} - M) T_1 & (\gamma_2 + \gamma_1 M) T_2 \end{pmatrix} \tag{41}$$

The conservation of energy (when losses are ignored) is easy to prove, by decomposing the matrix $\mathcal{P}$ into successive matrices. As a matter of fact, one has:

$$\begin{aligned} &\frac{1}{2}(S_1+S_2)\ \mathrm{Re}\left[(\gamma_1 p_1 + \gamma_2 p_2)(\gamma_1 v_1^* + \gamma_2 v_2^*)\right. \\ &\qquad \left. + \gamma_1\gamma_2 (p_1 - p_2)(v_1^* - v_2^*)\right] = \\ &= \frac{1}{2}(S_1+S_2)\ \mathrm{Re}\,(\gamma_1 p_1 v_1^* + \gamma_2 p_2 v_2^*)\,. \end{aligned}$$

The first member of this equation defines the energy flux for the vector $\mathcal{V}$, and the second member for the vector $\tilde{\mathcal{V}}$. This equation is the expression of the conservation of energy for the matrices $\mathcal{T}_r$ or $\mathcal{T}_r^{-1}$. The conservation for the matrices $\mathcal{M}$ (if M is without losses) and $\mathcal{T}$ is evident, thus the conservation of energy for the matrix $\mathcal{P}$ is proved.

4.2. *Some examples of inhomogeneous lattices*

If the length between perforations is different for the two guides, the 2nd order transfer matrix T is different for the two guides: it would be the simplest case of inhomogeneous lattice. A more realistic case of an inhomogeneous lattice is the case where the temperature, i.e. ρ and c, is different in the two guides. Thus, in order to use the results of section 2, the vector $\mathcal{V}$ needs to be replaced by the following vector:

$$\begin{pmatrix} \gamma_1 \dfrac{p_1}{\rho_1} + \gamma_2 \dfrac{p_2}{\rho_2} \\ \gamma_1 v_1 + \gamma_2 v_2 \\ p_1 - p_2 \\ \rho_1 v_1 - \rho_2 v_2 \end{pmatrix}$$

For the pair of quantities $(p/\rho,\ v)$ or $(p,\ v\rho)$, the transfer matrix T is:

$$T = \begin{pmatrix} \cos k\ell & cj\sin k\ell \\ c^{-1} j \sin k\ell & \cos k\ell \end{pmatrix}$$

Nevertheless, this result is valid only when frequency tends to zero, and we prefer to consider simply the case of homogeneous temperature, but different transfer matrices T_1 and T_2 for the two guides: this difference can be due for example to discontinuities located between two successive perforations. The problem of regular, branched resonators with their cavities coupled by perforations is an example of this case.

Another example of inhomogeneous lattice is the case of two guides with different Mach numbers M_i , when flow exists. The exact calculation of the effect of the perforation is not possible within the pure acoustic theory. Thus we need to use the theory as given by Sullivan (1979), with an experimental value of the shunt admittance Y_s (see the review by Munjal, 1987). Nevertheless, on the one hand we add a series impedance Z_a, and on the other hand we assume that no flow exists in a perforation: it is necessary in order to have a periodic lattice, with uniform values of the Mach numbers in both guides.

In order to obtain the transfer matrix $\mathcal{P}$ for a cell, we use the aeroacoustic state variables (see e.g. Munjal, 1987) and it is easy to show that the result is given by the equation (41). In this equation, the transfer matrix T_i for a guide is:

$$T_i = \mathrm{e}^{-j\mathrm{M}_i k_i \ell} \begin{pmatrix} \cos k_i\ell & j\rho c \sin k_i\ell \\ j\rho^{-1}c^{-1}\sin k_i\ell & \cos k_i\ell \end{pmatrix} \qquad (42)$$

where $k_i = k/(1-\mathrm{M}_i^2)$.

The matrix M is given by the equation (2), Y_s corresponding to the result given by Munjal (1987). We notice that in the case of perforated tube mufflers, the value of Y_s increases with the Mach number in the internal tube, then for homogeneous lattice, the first cutoff frequency increases as well (see Eq. (39)).

4.3. *Derivation of the dispersion equation*

For a regular lattice, we are searching for the eigenvalues λ of the matrix $\mathcal{P}$ (Eq. (41)). A small rearrangement shows that they are solutions of:

$$\begin{vmatrix} T_1 - T_1' - \lambda & T_2' \\ T_1' & T_2 - T_2' - \lambda \end{vmatrix} = 0 \qquad (43)$$

where $T_1' = \gamma_2(1-M)T_1$ and $T_2' = \gamma_1(1-M)T_2$.

By using the fact that det $M = 1$, one obtains after rather tedious calculation:

$$\begin{aligned} &\lambda^4 - \lambda^3\left(\overline{T_1} - \overline{T_1'} + \overline{T_2} - \overline{T_2'}\right) \\ &+\lambda^2\Big[\left(\overline{T_1} - \overline{T_1'}\right)\left(\overline{T_2} - \overline{T_2'}\right) - \overline{T_1'}\,\overline{T_2'} \\ &\quad +\det(T_1 - T_1') + \det(T_2 - T_2')\Big] \\ &-\lambda\left[\left(\overline{T_1} - \overline{T_1'}\right)\det T_2 + \left(\overline{T_2} - \overline{T_2'}\right)\det T_1\right] \\ &\quad + \det T_1T_2 = 0 \end{aligned}$$

where the superscript $^-$ indicates the trace of a matrix.

By using $\gamma_1 + \gamma_2 = 1$, and the following theorem, valid for second order matrices:

$$\overline{T_1T_2} = \overline{T_1}\,\overline{T_2} - \det T_1 - \det T_2 + \det(T_1 - T_2),$$

one can rearrange the equation and obtain:

$$1 + \frac{\lambda\gamma_2\,\overline{(1\!\!1 - M)T_1}}{\lambda^2 - \overline{T_1}\,\lambda + \det T_1} + \frac{\lambda\gamma_1\,\overline{(1\!\!1 - M)T_2}}{\lambda^2 - \overline{T_2}\,\lambda + \det T_2} = \frac{\lambda^2\gamma_1\gamma_2 \det(1\!\!1 - M)(T_1 - T_2)}{\left(\lambda^2 - \overline{T_1}\,\lambda + \det T_1\right)\left(\lambda^2 - \overline{T_2}\,\lambda + \det T_2\right)}. \qquad (44)$$

One recognizes in the denominators the dispersion equation for each matrix T_1 and T_2. Two factors of the second member are easily interpreted: det $(1-M)$ "measures" the magnitude of the perforation effect (it vanishes for zero Z_a or Y_s), and $\det(T_1 - T_2)$ "measures" the inhomogeneity of the lattice.

The equation (44) has four solutions, in general rather complicated. In the case of no flow, det $T_1 =$ det $T_2 = 1$, and the four solutions are coupled two by two (λ', $1/\lambda'$, λ'', $1/\lambda''$), due to reciprocity: the propagation constants are identical in the two directions of propagation (by writting $\lambda + 1/\lambda = 2\cosh\Gamma$, Eq. (38) becomes of second order). We will now discuss some approximate or exact solutions corresponding to particular cases.

In the case of a homogeneous lattices ($T_1 = T_2 = T$), we obtain a generalization of the equation (33):

$$\left(\lambda^2 - \lambda\,\overline{MT} + \det T\right)\left(\lambda^2 - \lambda\,\overline{T} + \det T\right) = 0 \qquad (45)$$

The second factor gives the plane mode, the first one the flute mode: all the discussion on the flute mode is valid, by replacing the quantity $2\cosh\Gamma$ by $[\lambda e^{jMk\ell} + 1/\lambda e^{jMk\ell}$ (see Eq. (42)). In particular the discussion about the cutoff frequency is unchanged.

In the case of tubes with very different cross section areas (e.g. $S_1 \ll S_2$), the equation (44) can be rewritten as follows:

$$\begin{aligned} &\left[\lambda^2 - \overline{MT_1}\,\lambda + \det T_1 + \lambda\gamma_1\,\overline{(1\!\!1 - M)(T_2 - T_1)}\right] \\ &\qquad \times\left[\lambda^2 - \overline{T_2}\,\lambda + \det T_2\right] = \\ &\qquad = \lambda^2\gamma_1\Big[\overline{(1\!\!1 - M)T_2}\left(\overline{T_1} - \overline{T_2}\right) \\ &\qquad\quad + \gamma_2 \det(1\!\!1 - M)\det(T_1 - T_2)\Big] \\ &\qquad + \lambda\gamma_1\,\overline{(1\!\!1 - M)T_2}\,(\det T_1 - \det T_2). \end{aligned}$$

For weakly inhomogeneous lattice, because γ_1 is small, the solutions are very close to the plane mode solutions in the guide 2, and the flute mode in the guide 1.

Another form of the equation (44) is interesting for the determination of the cutoff frequencies. If we restrict our study to symmetrical matrices $T_i = \begin{pmatrix} A_i & B_i \\ C_i & A_i \end{pmatrix}$, we obtain:

$$\begin{aligned} &1 - \mathsf{Z}_a\left(\frac{\gamma_2 C_1}{L_1 - A_1} + \frac{\gamma_1 C_2}{L_2 - A_2}\right) - \mathsf{Y}_s\left(\frac{\gamma_2 B_1}{L_1 - A_1} + \frac{\gamma_1 B_2}{L_2 - A_2}\right) \\ &\qquad = \mathsf{Y}_s\mathsf{Z}_a\left[1 + 2\left(\frac{\gamma_2 A_1}{L_1 - A_1} + \frac{\gamma_1 A_2}{L_2 - A_2}\right)\right. \\ &\qquad \left. - \frac{\gamma_1\gamma_2(A_1 - A_2)^2}{(L_1 - A_1)(L_2 - A_2)} + \frac{\gamma_1\gamma_2(B_1 - B_2)(C_1 - C_2)}{(L_1 - A_1)(L_2 - A_2)}\right] \end{aligned}$$

where $L_1 = (\lambda^2 + \det T_1)/2\lambda$, $L_2 = (\lambda^2 + \det T_2)/2\lambda$.

The form (33a) and (33b) of equation (33) suggests to transform the previous equation into the following equation (after tedious calculations):

$$\left[1-2\mathsf{Z}_a\lambda\left(\frac{\gamma_2 C_1}{\lambda^2-2A_1\lambda+\det T_1}+\frac{\gamma_1 C_2}{\lambda^2-2A_2\lambda+\det T_2}\right)\right]$$
$$\times\left[1-2\mathsf{Y}_s\lambda\left(\frac{\gamma_2 B_1}{\lambda^2-2A_1\lambda+\det T_1}+\frac{\gamma_1 B_2}{\lambda^2-2A_2\lambda+\det T_2}\right)\right]$$
$$=\mathsf{Y}_s\mathsf{Z}_a\left[\gamma_2\frac{\lambda^2-\det T_1}{\lambda^2-2A_1\lambda+\det T_1}+\gamma_1\frac{\lambda^2-\det T_2}{\lambda^2-2A_2\lambda+\det T_2}\right]^2 \tag{46}$$

4.4. *Stop and pass bands without flow*

When no flow occurs, as explained above, the equation (38) becomes of second order, the unknown being $\cosh\Gamma=(\lambda+1/\lambda)/2$. The solution is general real, thus Γ is either real (stop bands) or imaginary (pass bands). The equation (45) shows that if $\lambda=\pm1$ ($\cosh\Gamma=\pm1$), the second member vanishes, thus the cutoff frequencies depend either from Z_a or for Y_s: it is a generalization of the result obtained in the homogeneous case. These frequencies are given by equations generalizing equations (38a) and (38b).

$$\pm1=\mathsf{Z}_a\left(\frac{\gamma_2 C_1}{1\mp A_1}+\frac{\gamma_1 C_2}{1\mp A_2}\right) \tag{47a}$$

or

$$\pm1=\mathsf{Y}_s\left(\frac{\gamma_2 B_1}{1\mp A_1}+\frac{\gamma_1 B_2}{1\mp A_2}\right) \tag{47b}$$

Nevertheless, $\cosh\Gamma$ can be complex. We were surprised by this result, because for a 1D lattice, complex eigenvalues are not compatible with conservation of energy. This is possible in the present case only if the energy flux in one guide is exactly opposite to the energy flux in the other guide, a very particular case of evanescent waves. We show that this case is actually possible, by restricting our discussion to a simple case: $\mathsf{Y}_s=0$ (the discussion would be similar in the dual case: $\mathsf{Z}_a=0$). Consider an eigenvector (p_1, v_1, p_2, v_2). From equation (43), we obtain the following result:

$$p_1=-\frac{A_1-\lambda}{C_1}v_1,\qquad p_2=-\frac{A_2-\lambda}{C_2}v_2 \tag{48}$$

$$\frac{v_1}{v_2}=\frac{-\mathsf{Z}_a\gamma_2 C_1}{\cosh\Gamma+A_1+\gamma_2\mathsf{Z}_a C_1}=\frac{\cosh\Gamma+A_2+\gamma_1\mathsf{Z}_a C_2}{-\mathsf{Z}_a\gamma_1 C_2}. \tag{49}$$

The equations (48) and (49) give again the dispersion equation for $\cosh\Gamma$. The solutions are found to be:

$$\cosh\Gamma=-\frac{1}{2}\left(A_1+A_2+\gamma_2\mathsf{Z}_a C_1+\gamma_1\mathsf{Z}_a C_2\pm\delta^{1/2}\right) \tag{50}$$

where

$$\delta=(A_1-A_2+\gamma_2\mathsf{Z}_a C_1-\gamma_1\mathsf{Z}_a C_2)^2+4\gamma_1\gamma_2\mathsf{Z}_a^2 C_1 C_2.$$

Because we consider no dissipative lattices, Z_a, C_1, C_2 are imaginary quantities, and A_1 and A_2 are real quantities. Thus a necessary condition for having a complex solution is:

$$\mathsf{Z}_a^2 C_1 C_2<0 \tag{51}$$

(at low frequencies, if C_1 and C_2 are capacitive, this condition cannot be satisfied). Now the energy flux I is given by:

$$2I(S_1+S_2)=\gamma_1\,\mathrm{Re}\,(p_1v_1^*)+\gamma_2\,\mathrm{Re}\,(p_2v_2^*)$$

or $$2I(S_1+S_2)=\gamma_1\,\mathrm{Re}\left(\frac{\lambda}{C_1}\right)|v_1|^2+\gamma_2\,\mathrm{Re}\left(\frac{\lambda}{C_2}\right)|v_2|^2.$$

From equation (42), we obtain:

$$\frac{2I(S_1+S_2)}{|v_1v_2|}=\gamma_1\,\mathrm{Re}\left(\frac{\lambda}{C_1}\right)\left|\frac{\mathsf{Z}_a\gamma_2 C_1}{\cosh\Gamma+A_1+\gamma_2\mathsf{Z}_a C_1}\right|+\gamma_2\,\mathrm{Re}\left(\frac{\lambda}{C_2}\right)\left|\frac{\mathsf{Z}_a\gamma_2 C_2}{\cosh\Gamma+A_2+\gamma_1\mathsf{Z}_a C_1}\right|.$$

$\mathsf{Z}_a C_1$ and $\mathsf{Z}_a C_2$ being real quantities, we write $|\mathsf{Z}_a C_1|=\varepsilon\mathsf{Z}_a C_1$, and $|\mathsf{Z}_a C_2|=-\varepsilon\mathsf{Z}_a C_2$, where $\varepsilon=$ sign ($\mathsf{Z}_a C_1$) and the inequality (51) is assumed to be satisfied. Thus we obtain the following result:

$$\frac{2I(S_1+S_2)}{|v_1v_2|}=\gamma_1\gamma_2\,\mathrm{Re}\left[\lambda\mathsf{Z}_a\varepsilon\left(\frac{1}{|\cosh\Gamma+A_1+\gamma_2\mathsf{Z}_a C_1|}-\frac{1}{|\cosh\Gamma+A_2+\gamma_1\mathsf{Z}_a C_2|}\right)\right].$$

If the discriminant δ (Eq. (50)) is negative, the last factor is zero, and the energy flux can be zero with complex value for λ. In this case, the energy flux in the two guides is opposite in sign, and exponentially decreasing from one cell to another, because the modulus of λ is less than unity.

As a summary, there are few cases for a lattice without flow:

- a superposition of two propagating waves
- a superposition of one propagating and one evanescent wave
- a superposition of two evanescent waves
- a superposition of two, non classical, evanescent waves, with non-zero energy flux in each guide, such that the propagation constants of the two waves are complex conjugates.

(The two first cases are the only possible cases for homogeneous lattices).

4.5. *Weakly inhomogeneous lattice with flow*

In the case where the two guides are with flow but small Mach number, the matrices T_1 and T_2 are not very different. Thus it is interesting to use a perturbation method in order to deduce the solutions, starting from the solutions of the equation (45). If one writes the eigenvalues in each guide $\lambda_1^\pm$, $\lambda^\pm$, with:

$$\lambda_1^\pm = \lambda^\pm + \varepsilon^\pm, \quad \lambda_2^\pm = \lambda^\pm - \varepsilon^\pm,$$

the transfer matrix T_1 is:

$$T_1 = \frac{1}{2}\begin{pmatrix} \lambda_1^+ + \lambda_1^- & \lambda_1^+ - \lambda_1^- \\ \lambda_1^+ - \lambda_1^- & \lambda_1^+ + \lambda_1^- \end{pmatrix}$$

and the expression for T_2 is similar. The solutions of the equation (45) are $\lambda^\pm$ for the second factor, and $\lambda_p^\pm$ for the first factor. In the case of a weakly inhomogeneous lattice, the eigenvalues are close to $\lambda^\pm$ and $\lambda_p^\pm$. We restrict the following calculation to the eigenvalues close to $\lambda^\pm$, by proving that they are independent of the perforations.

By using the form (46) of the dispersion equation, we obtain:

$$\left[1 - \frac{\mathsf{Z}_a}{\rho_c}(Q_1 + Q_2)\right]\left[1 - \mathsf{Y}_s \rho c (Q_1 + Q_2)\right] = \\ = \mathsf{Y}_s \mathsf{Z}_a \left[R_1 + R_2\right]^2$$

where

$$Q_1 = \lambda\gamma_2\left[\left(\lambda - \lambda_1^+\right)^{-1} - \left(\lambda - \lambda_1^-\right)^{-1}\right]$$

$$R_1 = \gamma_2\left(\lambda_1^+ - \lambda_1^-\right)^{-1}\left(\lambda_1^+ \frac{\lambda - \lambda_1^-}{\lambda - \lambda_1^+} - \lambda_1^- \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}\right).$$

The expressions for Q_2 and R_2 are similar. We are searching the first solution as $\lambda^+ + x$, where x is of the order of $\varepsilon^+ \cdot Q_1$, Q_2, R_1, R_2 are expanded with respect to ε^+ and x, e.g.:

$$Q_1 = Q_1^{(-1)} + Q_1^{(0)} + \ldots$$

Because $Q_i^{(-1)} = R_i^{(-1)}$, the term of the order (-2) in ε^+ vanishes identically, and in order to have zero term of the order (-1), x needs to be solution of the following equation:

$$Q_1^{(-1)} + Q_2^{(-1)} = 0$$

thus $$\gamma_2/(x - \varepsilon^+) + \gamma_1(x + \varepsilon^+) = 0, \tag{52}$$
and $$\lambda = \lambda^+ + x = \gamma_1\lambda_1^+ + \gamma_2\lambda_2^+.$$

Similarly, another solution is found to be:

$$\lambda = \gamma_1\lambda_1^- + \gamma_2\lambda_2^-.$$

At the first order of the Mach number M_1 and M_2 and for small $k\ell$, these solutions are the eigenvalues corresponding to a plane wave with Mach number: $\mathrm{M} = \gamma_1\mathrm{M}_1 + \gamma_2\mathrm{M}_2$. The plane mode of a weakly inhomogeneous lattice is therefore found to be the plane mode in a guide of section $S_1 + S_2$, and a flow volume velocity equal to the sum of the flow volume velocities in each guide. Unfortunately, the flute mode is more complicated to analyse.

5. Conclusion

Let us summarize the new results of the present paper:

- The impedance of a perforation can be calculated by using the modal theory, but it is not sufficient for describing the whole effect when excitation is not symmetrical. The existence of an antisymmetrical impedance Z_a is proved, but it was known for similar problems of electromagnetic waveguides (see Marcuvitz, 1948). Exact values of the elements in simple geometries remain to compute. The modal theory gives the result in a form directly useful for analysing homogeneous lattices.
- For homogeneous lattices, either regular or irregular, there are two modes: the plane one and the "flute" one. In a periodic lattice, the flute mode can be either propagating or evanescent, the cutoff frequencies being dependent on only one element of the equivalent circuit, either the shunt admittance or the series impedance. In general, the first cutoff depends on the shunt admittance, and it is an important quantity characterizing a perforated tube muffler. In stop bands, it is often possible to find a length correction for describing the junction between two guides at the input of a lattice constituted by the same guides provided with perforations.
- For inhomogeneous lattices (if propagation is different in the two guides), when no flow occurs, the cutoff frequencies are dependent on only one element of the equivalent circuit. Very particular stop bands can exist, the energy flux in each guide is non zero, but opposite in sign, thus the total flux is zero. Flux through perforations allows the exponential decreasing of the flux in each guide. When weak flow occurs, the lattice is quasi-homogeneous and the plane mode corresponds to a guide with a flow volume velocity being the sum of the flow volume velocities in each guide.

Basically this theory is valid for far located perforations in order to avoid coupling by evanescent modes of the guides. The condition is known to be that the distance between perforation is equal or larger than the transverse dimensions of the guides. This condition can be severe for some practical situations, and can justify the classical continuous model. Nevertheless, the present theory allows to obtain a complete description for the studied problem, and it could be improved by taking into account the interaction between perforations. Moreover, we recognize that the main interest of our approach concerns guides without flow, but often the concepts are not destroyed by weak flow. Finally, we will emphasize the fact that we use the specific acoustic impedance rather the acoustic impedance, and for certain use of our results, it is necessary to rewrite them in another form.

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